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MATHEMATICS
TARGET IIT JEE

MATRICES & DETERMINANTS

THEORY AND EXERCISE BOOKLET

CONTENTS

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JEE Syllabus :

Matrices as a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of a square matrix of order up to three, inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables

A. DEFINITION

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix.** If a matrix has **m** rows and **n** columns then the **order** of matrix is said to be **m** by **n** (denoted as **m** × **n**).

The general **m** × **n** matrix is $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots & a_{ij} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}$

where a_{ij} denote the element of **i^{th} row & j^{th} column**. The above matrix is usually denoted as $[a_{ij}]_{m \times n}$

Note :

(i) The elements a_{11}, a_{22}, a_{33},....... are called as **diagonal elements.** Their sum is called as **trace of A** denoted as tr(A)

(ii) Capital letters of English alphabets are used to denote matrices.

B. TYPES OF MATRICES

(i) Row Matrix : A matrix having only one row is called as row matrix (or row vector).
General form of row matrix is $A = [a_{11}, a_{12}, a_{13}, \ldots\ldots, a_{1m}]$

(ii) Column Matrix : A matrix having only one column is called as column matrix (or column vector). General form of $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix}$

(iii) Square Matrix : A matrix in which number of rows & columns are equal is called a square matrix.

The general form of a square matrix is $A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{bmatrix}$ we denote as $A = [a_{ij}]_n$.

(iv) Zero Matrix : $A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0 \ \forall$ i & j.

(v) Upper Triangular Matrix : $A = [a_{ij}]_{m \times n}$ is said to be upper triangular matrix, if $a_{ij} = 0$ for $i > j$ (i.e., all the elements below the diagonal element are zero.)

(vi) Lower Triangular Matrix : $A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for $i < j$ (i.e., all the elements above the diagonal elements are zero.)

(vii) Diagonal matrix : A square matrix $[a_{ij}]_{m \times n}$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$ (i.e., all the elements of the square matrix other than diagonal elements are zero)
Note : Diagonal matrix of order in is denoted as Diag $(a_{11}, a_{22}, \ldots\ldots\, a_{nn})$.

(viii) Scalar Matrix : Scalar matrix is a diagonal matrix in which all the diagonal elements are same.
$A = [a_{ij}]_n$ is a scalar matrix, if (i) $a_{ij} = 0$ for $i \neq j$ and (ii) $a_{ij} = k$ for $i = j$.

(ix) Unit Matrix (Identity Matrix) : Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted y I_n (or I).

i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ij} = 1$ eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

C. OPERATIONS ON MATRICES

(i) Equality of Matrices : Two matrices A ad B are said to be equal if they are comparable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$. A = B if (i) m = p, n = q (ii) $a_{ij} = b_{ij} \forall$ i & j.

(ii) Addition of Matrices : Let A and B be two matrices of same order (i.e. comparable matrices). Then A + B is defined to be $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij} \forall$ i & j.

(iii) Substraction of Matrices : Let A & B be two matrices of same order. Then A – B is defined as A + (–B) where – B is (–1) B.

(iv) Multiplication of Matrix By Scalar : Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij} \ \forall$ i & j.

Note : If A is a scalar matrix, then $A = \lambda I$, where λ is the diagonal element.

(v) Properties of Addition & Scalar Multiplication : Consider all matrices of order m × n, whose elements are from a set F (F denote Q, R or C).

Let $M_{m\times n}$ (F) denote the set of all such matrices. Then

(a) $A \in M_{m \times n}(F)$ & $B \in M_{m \times n}(F) \Rightarrow A + B \in M_{m \times n}(F)$

(b) A + B = B + A

(c) (A + B) + C = A + (B + C)

(d) $O = [o]_{m \times n}$ is the additive identity.

(e) For every $A \in M_{n \times m}(F)$, – A is the additive inverse.

(f) $\lambda (A + B) = \lambda A + \lambda B$

(g) $\lambda A = A\lambda$

(h) $(\lambda_1 + \lambda_2) A = \lambda_1 A + \lambda_2 A$

Ex.1 For the following pairs of matrices, determine the sum and difference, if they exist.

(a) $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1.5 & 6 \\ -3 & 2+i & 0 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \end{pmatrix}$

Sol. **(a)** Matrices A and B are 2 × 3 and confirmable for addition and subtraction.

$$A + B = \begin{pmatrix} 1+2 & -1+1.5 & 2+6 \\ 0-3 & 1+2+i & 3+0 \end{pmatrix} = \begin{pmatrix} 3 & 0.5 & 8 \\ -3 & 3+i & 3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1-2 & -1-1.5 & 2-6 \\ 0-(-3) & 1-(2+i) & 3-0 \end{pmatrix} = \begin{pmatrix} -1 & -2.5 & -4 \\ 3 & -1-i & 3 \end{pmatrix}$$

(b) Matrix A is 2 × 2, and B is 2 × 3. Since A and B are not the same size, they are not confirmable for addition or subtraction.

Ex.2 Find the additive inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & -1 & 2 & 2 \\ 1 & 2 & 8 & 7 \end{bmatrix}$.

Sol. The additive inverse of the 3 × 4 matrix A is the 3 × 4 matrix each of whose elements is the negative of the corresponding element of A. Therefore if we denote the additive inverse of A by – A, we have

$-A = \begin{bmatrix} -2 & -3 & 1 & -1 \\ -3 & 1 & -2 & -2 \\ -1 & -2 & -8 & -7 \end{bmatrix}$. Obviously A + (–A) = (–A) + A = O, where O is the null matrix of the type 3 × 4.

Ex.3 If $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$, find the matrix D such that A + B – D = 0.

Sol. We have $A + B - D = 0 \Rightarrow (A + B) + (-D) = 0 \Rightarrow A + B = (-D) = D \quad \therefore\ D = A + B = \begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$.

Ex.4 If $A = \begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix}$, verify that 3(A + B) = 3A + 3B.

Sol. We have $A + B = \begin{bmatrix} 3+4 & 9+0 & 0+2 \\ 1+7 & 8+1 & -2+4 \\ 7+2 & 5+2 & 4+6 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 2 \\ 8 & 9 & 2 \\ 9 & 7 & 10 \end{bmatrix}$ $\therefore\ 3(A + B) = \begin{bmatrix} 3\times7 & 3\times9 & 3\times2 \\ 3\times8 & 3\times9 & 3\times2 \\ 3\times9 & 3\times7 & 3\times10 \end{bmatrix} = \begin{bmatrix} 21 & 27 & 6 \\ 24 & 27 & 6 \\ 27 & 21 & 30 \end{bmatrix}$

Again $3A = 3\begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 3\times3 & 3\times9 & 3\times0 \\ 3\times1 & 3\times8 & 3\times-2 \\ 3\times7 & 3\times5 & 3\times4 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix}$

Also $3B = 3\begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 3\times4 & 3\times0 & 3\times2 \\ 3\times7 & 3\times1 & 3\times4 \\ 3\times2 & 3\times2 & 3\times6 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix}$

$\therefore\quad 3A + 3B = \begin{bmatrix} 9 & 27 & 0 \\ 3 & 24 & -6 \\ 21 & 15 & 12 \end{bmatrix} + \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix} \Rightarrow \begin{bmatrix} 9+12 & 27+0 & 0+6 \\ 3+21 & 24+3 & -6+12 \\ 21+6 & 15+6 & 12+18 \end{bmatrix} = \begin{bmatrix} 21 & 27 & 6 \\ 24 & 27 & 6 \\ 27 & 21 & 30 \end{bmatrix}$

$\therefore$ 3 (A + B) = 3A + 3B, i.e. the scalar multiplication of matrices distributes over the addition of matrices.

Ex.5 The set of natural numbers N is partitioned into arrays of rows and columns in the form of matrices as

$M_1 = (1)$, $M_2 = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, $M_3 \begin{pmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{pmatrix}$,......., M_n = ( ) and so on. Find the sum of the elements of the diagonal in M_n.

Sol. Let $M_n = (a_{ij})$ where i, j = 1, 2, 3,.........,n.

We first find out a_{11} for the n^{th} matrix; which is the n^{th} term in the series ; 1, 2, 6,......

Let $S = 1 + 2 + 6 + 15 + \ldots + T_{n-1} + T_n$. Again writing $S = 1 + 2 + 6 + \ldots + T_{n-1} + T_n$

$\Rightarrow\quad 0 = 1 + 1 + 4 + 9 + \ldots + (T_n - T_{n-1}) - T_n \quad\Rightarrow\quad T_n = 1 + (1 + 4 + 9 + \ldots$ upto (n – 1) terms)

$= 1 + (1^2 + 2^2 + 3^2 + 4^2 + \ldots + (n-1)^2) = 1 + \dfrac{n(n-1)(2n-1)}{6}$.

Now, observing carefully, the consecutive distance between the elements of the diagonal of the n^{rh} matrix is n + 1.

Therefore first term is $1 + \dfrac{n(n-1)(2n-1)}{6}$ and common difference = n + 1.

Hence the required sum $M_n = \dfrac{n}{2}\left[2\left(1 + \dfrac{n(n-1)(2n-1)}{6}\right) + (n-1)(n+1)\right]$

$= \dfrac{n}{6}\ [6 + (n-1)(2n^2 + 2n + 3)] = \dfrac{n}{6}\ [2n^3 + n + 3)]$.

(vi) Multiplication of Matrices : Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$. Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}a_{kj}$ which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

Note :

1. The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined $\not\Rightarrow$ BA is defined.

2. In general $AB \neq BA$, even when both the products are defined.

3. A(BC) = (AB) C, whenever it is defined.

(vii) Properties of Matrix Multiplication : Consider all square matrices of order 'n'. Let M_n (F) denote the set of all square matrices of order n, (where F is Q, R or C). Then

(a) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(b) In general $AB \neq BA$

(c) (AB) C = A(BC)

(d) I_n, the identity matrix of order n, is the multiplicative identity. $AI_n = A = I_nA \quad \forall A \in M_n(F)$

(e) For every non singular matrix A(i.e., $|A| \neq 0$) of M_n (F) there exist a unique (particular) matrix $B \in M_n(F)$ so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar $(\lambda A) B = \lambda(AB) = A(\lambda B)$.

(g) $A(B + C) = AB + AC \quad \forall A, B, C \in M_n(F)$

(h) $(A + B)C = AC + BC \quad \forall A, B, C \in M_n(F)$

Note :

1. Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

2. For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Ex.6 If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, find AB and BA and show that $AB \neq BA$.

Sol. We have $AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2.1-3.1+4.0 & 2.3+3.2+4.0 & 2.0+3.1+4.2 \\ 1.1-2.1+3.0 & 1.3+2.2+3.0 & 1.0+2.1+3.2 \\ -1.1-1.1+2.0 & -1.3+1.2+2.0 & -1.0+1.1+2.2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$

Similarly, $BA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1.2+3.1-0.1 & 1.3+3.2+0.1 & 1.4+3.3+0.2 \\ -1.2+2.1-1.1 & -1.3+2.2+1.1 & -1.4+2.3+1.2 \\ 0.2+0.1-2.1 & 0.3+0.2+2.1 & 0.4+0.3+2.2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$

The matrix AB is of the type 3 × 3 and the matrix BA is also of the type 3 × 3. But the corresponding elements of these matrices are not equal. Hence $AB \neq BA$.

Ex.7 Show that for all values of p, q, r, s the matrices, $P = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$, and $Q = \begin{bmatrix} r & s \\ -s & r \end{bmatrix}$, PQ = QP.

Sol. We have $PQ = \begin{bmatrix} pr-qs & ps+qr \\ -qr-ps & -qs+pr \end{bmatrix}$.

Also $QP = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}\begin{bmatrix} p & q \\ -q & p \end{bmatrix} = \begin{bmatrix} rp-sq & rq+sp \\ -sp-rq & -sq+rp \end{bmatrix} = \begin{bmatrix} pr-qs & ps+pq \\ -qr-ps & -qs+pr \end{bmatrix}$ for all values of p, q, r, s.

Hence PQ = QP, for all values of p, q, r, s.

Ex.8 If A = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ and C = $\begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$ show that AB = AC though B ≠ C.

Sol. We have AB = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1.1-3.2+2.1 & 1.4+3.1-2.2 & 1.1-3.1+2.1 & 1.0-3.1+2.2 \\ 2.1+1.2-3.1 & 2.4+1.1+3.2 & 2.1+1.1-3.1 & 2.0+1.1-3.2 \\ 4.1-3.2-1.1 & 4.4-3.1+1.2 & 4.1-3.1-1.1 & 4.0-3.1-1.2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}.$$

Also AC = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$ $\quad\therefore$ AB = AC, though B ≠ C.

Ex.9 If A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$, where k is any positive integer.

Sol. We shall prove the result by induction on k.

We have $A_1 = A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix}$. Thus the result is true when k = 1.

Now suppose that the result is true for any positive integer k.

i.e., $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ where k is any positive integer.

Now we shall show that the result is true for k + 1 if it is true for k. We have

$$A^{K+1} = AA^k = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} = \begin{bmatrix} 3+6k-4k & -12k-4+8k \\ 1+2k-k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+2k & -4-4k \\ 1+k & -2k-1 \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(1+k) \\ 1+k & 1-2(1+k) \end{bmatrix}.$$

Thus the result is true for k + 1 if it is true for k. But it is true for k = 1. Hence by induction it is true for all positive integral value of k.

Ex.10 Find real numbers c_1 and c_2 so that $I + c_1M + c_2M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ where $M = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and I is the identity matrix.

Sol. $M^2 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 4 \end{bmatrix}$; $I + c_1M + c_2M^2 = \begin{bmatrix} 1+c_1+c_2 & 3c_1+9c_2 \\ 0 & 1+2c_1+4c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $c_1 + c_2 = -1$ and $3(c_1 + c_2) + 6c_2 = 0$ $\Rightarrow$ $c_2 = 1/2$, $c_1 = -3/2$

Ex.11 If A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, prove that $(aI + bA)^n = a^n I + na^{n-1} bA$. for " a, b ∈ R where I is the two rowed unit matrix n is a positive integer.

Sol. A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ $A^2 = A \,.\, A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$ $\Rightarrow$ $A^3 = A^2 \,.\, A = 0$ $\Rightarrow$ $A^2 = A^3 = A^4 = \ldots\ldots A^n = 0$

Now by binomial theorem

$(a\,I + b\,A)^n = (a\,I)^n + {}^nC_1(a\,I)^{n-1}\, b\,A + {}^nC_2\,(a\,I)^{n-2}\,(b\,A)^2 + \ldots.. + {}^nC_n\,(b\,A)^n$

$= a^n\,I + {}^nC_1\,a^{n-1}\,b\,I\,A + {}^nC_2\,a^{n-2}\,b^2\,I\,A^2 + \ldots\ldots + {}^nC_n\,b^n\,A^n$

$= a^n\,I + n\,a^{n-1}\,b\,A + 0\ldots\ldots$ $\quad(\because A^n = 0)$ $\Rightarrow$ $(a\,I + b\,A)^n = a^n\,I + n\,a^{n-1}\,b\,A.$

Ex.12 If $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 18 & 2007 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{bmatrix}$ then find the value of (n + a).

Sol. Consider $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2a+8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3a+24 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n & na+8\sum\limits_{k=0}^{n-1} k \\ 0 & 1 & 4n \\ 0 & 0 & 1 \end{bmatrix}^n$ Hence n = 9 and 2007 = 9a + $8\sum\limits_{k=0}^{8} k$ = 9a + $8\left(\dfrac{8\cdot 9}{2}\right)$

$\Rightarrow$ 2007 = 9a + 32 · 9 = 9(a + 32) $\Rightarrow$ a + 32 = 223 $\Rightarrow$ a = 191 hence a + n = 200

Ex.13 Find the matrices of transformations T_1T_2 and T_2T_1, when T_1 is rotation through an angle 60º and T_2 is the reflection in the y–axis. Also verify that $T_1T_2 \neq T_2T_1$.

Sol. $T_1 = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore T_1T_2 = \dfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} -1+0 & 0-\sqrt{3} \\ -\sqrt{3}+0 & 0+1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ ....(1)

and $T_2T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \dfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} -1+0 & \sqrt{3}+0 \\ 0+\sqrt{3} & 0+1 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$ .... (2)

It is clear from (1) and (2), $T_1T_2 \neq T_2T_1$

Ex.14 Find the possible square roots of the two rowed unit matrix I.

Sol. Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be square root of the matrix I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then A^2 = I.

i.e. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ i.e. $\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since the above matrices are equal, therefore

$a^2 + bc = 1$ ...(i) $ac + cd = 0$ ...(iii)

$ab + bd = 0$ ...(ii) $cb + d^2 = 0$ ...(iv) must hold simultaneously.

If a + d = 0, the above four equations hold simultaneously if d = –a and $a^2 + bc = 1$.

Hence one possible square root of I is

A = $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ where α, β, γ are any three numbers related by the condition $\alpha^2 + \beta\gamma = 1$.

If a + d ≠ 0, the above four equations hold simultaneously if b = 0, c = 0, a = 1, d = 1 or if b = 0, c = 0, a = –1, d = –1. Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ i.e. ± I are other possible square roots of I.

Ex.15 If $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $B = \begin{bmatrix} x & -x \\ -x & x \end{bmatrix}$, then prove that $x\, e^A = \frac{1}{2}(A \,.\, e^{2x} + B)$. (where $e^A = I + A + \frac{A^2}{2!} + \ldots\ldots$)

Sol. We have $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = x\,E$ ...(1) where $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 = A \,.\, A = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} . \, x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = x^2\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2\,x^2\,E$ ...(2)

$A^3 = A^2 \,.\, A = 2\,x^2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} . \, x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2^2\,x^3\,E$ ...(3)

Similarly it can be shown that $A^4 = 2^3\,x^4\,E$, $A^5 = 2^4\,x^5\,E$ .....

Now, $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots = I + x \,.\, E + \frac{2x^2E}{2!} + \frac{2^2x^3E}{3!} + \ldots$ [by (1), (2), (3)]

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{2x^2}{2!}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \ldots = \begin{bmatrix} 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\ldots & x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\ldots \\ x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\ldots & 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\ldots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\ldots\right)+\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\ldots\right)-\frac{1}{2} \\ \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\ldots\right)-\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\ldots\right)+\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (e^{2x}+1) & (e^{2x}-1) \\ (e^{2x}-1) & (e^{2x}+1) \end{bmatrix} = \frac{1}{2}e^{2x}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow e^A = \frac{1}{2}\left(e^{2x}\frac{A}{x}+\frac{B}{x}\right) \Rightarrow x \,.\, e^A = \frac{1}{2}(e^{2x} \,.\, A + B)$$

D. FURTHER TYPES OF MATRICES

(a) Nilpotent matrix : A square matrix A is said to be nilpotent (of order 2) if, $A^2 = O$.
A square matrix is said to be nilpotent of order p, if p is the least positive integer such that $A^p = O$

(b) Idempotent matrix : A square matrix A is said to be idempotent if, $A^2 = A$.

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

(c) Involutory matrix : A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

(d) Orthogonal matrix : A square matrix A is said to be an orthogonal matrix if $A'A = I = A'A$

(e) Unitary matrix : A square matrix A is said to be unitary if $A(\overline{A})' = I$, where $\overline{A}$ is the complex conjugate of A.

Ex.16 Find the number of idempotent diagonal matrices of order n.

Sol. Let A = diag (d_1, d_2,....., d_n) be any diagonal matrix of order n.

now $A^2 = A\,.\,A = \begin{bmatrix} d_1 & 0 & 0 & .... & 0 \\ 0 & d_2 & 0 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & .... & d_n \end{bmatrix} \times \begin{bmatrix} d_1 & 0 & 0 & .... & 0 \\ 0 & d_2 & 0 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & .... & d_n \end{bmatrix} = \begin{bmatrix} d_1^2 & 0 & 0 & .... & 0 \\ 0 & d_2^2 & 0 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & .... & d_n^2 \end{bmatrix}$

But A is idempotent, so $A^2 = A$ and hence corresponding elements of A^2 and A should be equal

$\therefore$ $d_1^2 = d_1, d_2^2 = d_2, ..., d_n^2 = d_n$ or d_1 = 0, 1; d_2 = 0, 1;.........;d_n = 0, 1

$\Rightarrow$ each of d_1, d_2 ......, d_n can be filled by 0 or 1 in two ways.

$\Rightarrow$ Total number of ways of selecting d_1, d_2,......., d_n = 2^n

Hence total number of such matrices = 2^n.

Ex.17 Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent and find its index.

Sol. We have $A^2 = AA \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$

Again $A^3 = AA^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$

Thus 3 is the least positive integer such that $A^3 = 0$. Hence the matrix A is nilpotent of index 3.

Ex.18 If AB = A and BA = B then B′A′ = A′ and A′B′ = B′ and hence prove that A′ and B′ are idempotent.

Sol. We have AB = A $\Rightarrow$ (AB)′ = A′ $\Rightarrow$ B′A′ = A′. Also BA = B $\Rightarrow$ (BA)′ = B′ $\Rightarrow$ A′B′ = B′.

Now A′ is idempotent if A'^2 = A′. We have A'^2 = A′A′ = A′ (B′A′) = (A′B′) A′ = B′A′ = A′.

$\therefore$ A′ is idempotent.

Again B'^2 = B′B′ = B′ (A′B′) = (B′A′) B′ = A′B′ = B′. $\therefore$ B′ is idempotent.

Ex.19 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \left(a_{ij}(n)\right)$. If $\lim\limits_{n\to\infty} \dfrac{a_{12}(n)}{a_{22}(n)} = \ell$ where $\ell^2 = \sqrt{a} + \sqrt{b}$ (a, b $\in$ N), find the value of (a + b).

Sol. Suppose $A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = I + B$ (say)

hence $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = (I + B)^n$ $\therefore$ $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = {}^nC_0 I + {}^nC_1 B + {}^nC_2 B^2 + {}^nC_3 B^3 + {}^nC_4 B^4 + ...(1)$

now $B^2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$ Hence $B^{2k} = 2^k I$ and $B^{2k+1} = B^{2k}B = 2^k B$

now $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \underbrace{({}^nC_0 + {}^nC_2 \cdot 2 + {}^nC_4 \cdot 2^2 + ...)}_{'X'\text{ say}} I + \underbrace{({}^nC_1 + {}^nC_3 \cdot 2 + {}^nC_5 \cdot 2^2 + ...)}_{'Y'\text{ say}} B$

$\therefore$ $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} Y & Y \\ Y & -Y \end{bmatrix} = \begin{bmatrix} X+Y & Y \\ Y & X-Y \end{bmatrix}$

Hence a_{12} in $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = Y \quad \therefore \quad a_{12} = {}^nC_1 + {}^nC_3 \cdot 2 + {}^nC_5 \cdot 2^2 + {}^nC_7 \cdot 2^3 + \ldots\ldots..$

$$= \frac{1}{\sqrt{2}}\left[{}^nC_1 \cdot \sqrt{2} + {}^nC_3 \cdot (\sqrt{2})^3 + {}^nC_5 \cdot (\sqrt{2})^5 + \ldots\ldots..\right] = \frac{1}{\sqrt{2}}\left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2}\right]$$

|||ly $a_{22} = X - Y = ({}^nC_0 + {}^nC_2 \cdot 2 + {}^nC_4 \cdot 2^2 + {}^nC_6 \cdot 2^3 + \ldots.) - ({}^nC_1 + {}^nC_3 \cdot 2 + {}^nC_5 \cdot 2^2 + {}^nC_7 \cdot 2^3 + \ldots)$

$$= \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} - \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} = \frac{\sqrt{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n] - [(1+\sqrt{2})^n - (1-\sqrt{2})^n]}{2\sqrt{2}}$$

$$a_{22} = \frac{(\sqrt{2}-1)(1+\sqrt{2})^n - (\sqrt{2}+1)(1-\sqrt{2})^n}{2\sqrt{2}}$$

$$\therefore \lim_{n\to\infty} \frac{a_{12}}{a_{22}} = \lim_{n\to\infty} \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{(\sqrt{2}-1)(1+\sqrt{2})^n + (\sqrt{2}+1)(1-\sqrt{2})^n} = \lim_{n\to\infty} \frac{1-\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n}{(\sqrt{2}-1)+(\sqrt{2}+1)\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n} = \frac{1-0}{\sqrt{2}-1} = 1+\sqrt{2};$$

Hence $\ell^2 = (1+\sqrt{2})^2 = 3 + 2\sqrt{2} = \sqrt{9} + \sqrt{8}$. Hence $a + b = 9 + 8 = 17$.

E. TRANSPOSE OF MATRIX

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A'(or A^T) and is defined as $A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji} \forall$ i & j.

i.e. A' is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

(i) For any matrix $A = [a_{ij}]_{m \times n}$, $(A')' = A$

(ii) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(iv) $(A_1 \pm A_2 \pm \ldots. \pm A_n)' = A_1' \pm A_2' \pm \ldots.. \pm A_n'$, where A_j are comparable.

(v) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(vi) $(A_1 A_2 \ldots\ldots\ldots A_n)' = A_n'. A_{n-1}'\ldots\ldots\ldots\ldots\ldots\ldots \pm A_2'. A_1'$, provided the product is defined.

(vii) Symmetric & Skew–Symmetric Matrix : A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ij} \forall$ i & j.

A square matrix A is said to be skew–symmetric if $A' = -A$

i.e. Let $A = [a_{ij}]_n$. A is skew–symmetric iff $a_{ij} = -a_{ji} \forall$ i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix. & $B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew–symmetric matrix.

Note :

1. In skew–symmetric matrix all the diagonal elements are zero. ($\because a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$)

2. For any square matrix A, $A + A'$ is symmetric & $A - A'$ is skew – symmetric.

3. Every square matrix can be uniquely expressed as a sum of two square matrices of which one is symmetric and the other is skew–symmetric.

$A = B + C$, where $B = \frac{1}{2}(A + A')$ & $C = \frac{1}{2}(A - A')$

F. DETERMINANT

(i) Submatrix : Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

eg. $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$ Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}, \begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}, \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ are all submatrices of A.

(ii) Determinant of A Square Matrix :

Let $A\ [a]_{1\times 1}$ be a 1×1 matrix. Determinant A is defined as $|A| = a$ eg. $A = [-3]_{1\times 1}$ $|A| = -3$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|$ is defined as $ad - bc$. eg. $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}$, $|A| = 23$

(iii) Minors & Cofactors : Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij} is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

eg. $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow M_{11} = s = C_{11}$; $M_{12} = c, C_{12} = -c$; $M_{21} = b, C_{21} = -b$; $M_{22} = a = C_{22}$

eg. $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} p & r \\ y & z \end{vmatrix} = qz - yr = C_{11}$;

$$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx) = bx - ay \text{ etc.}$$

(iv) Determinant : Let $A = [a_{ij}]_n$ be a square matrix ($n > 1$). Determinant of A is defined as the sum of products of elements of any one row (or one column) with corresponding cofactors.

eg. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \text{ (using first row)} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \text{ (using second column)} = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

G. PROPERTIES OF DETERMINANTS

P–1 : The value of a determinant remains unaltered , if the rows & columns are inter changed . e.g. If

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$ $\Rightarrow$ D & D′ are transpose of each other.

If $D' = -D$ then it is **SKEW SYMMETRIC** determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0$

$\Rightarrow$ Skew symmetric determinant of third order has the value zero .

P– 2 : If any two rows (or columns) of a determinant be interchanged , the value of determinant is changed in sign only . e.g. Let D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & D′ = $\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then D′ = – D .

P– 3 : If a determinant has any two rows (or columns) identical , then its value is zero.

e.g. Let D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then it can be verified that D = 0 .

P–4 : If all the elements of any row (or column) be multiplied by the same number then the determinant is multiplied by that number. e.g. If D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & D′ = $\begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then D′= KD

P–5 : If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants.

e.g. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

P– 6 : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

e.g. Let D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and D′ = $\begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. Then D′ = D .

Note that while applying this property atleast one row (or column) must remain unchanged.

P– 7 : If by putting x = a the value of a determinant vanishes then (x – a) is a factor of the determinant

Ex.20 Find the value of the determinant $\begin{vmatrix} {}^nC_{r-1} & {}^nC_r & (r+1)^{n+2}C_{r+1} \\ {}^nC_r & {}^nC_{r+1} & (r+2)^{n+2}C_{r+2} \\ {}^nC_{r+1} & {}^nC_{r+2} & (r+3)^{n+2}C_{r+3} \end{vmatrix}$

Sol. Operating $C_1 \to C_1 + C_2$ and using ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$ in C_3, we get

$\begin{vmatrix} {}^{n+1}C_r & {}^nC_r & (n+2)^{n+1}C_r \\ {}^{n+1}C_{r+1} & {}^nC_{r+1} & (n+2)^{n+1}C_{r+1} \\ {}^{n+1}C_{r+2} & {}^nC_{r+2} & (n+2)^{n+1}C_{r+2} \end{vmatrix}$ = 0, as C_1 and C_3 are identical.

Ex.21 A is a n × n matrix (n > 2) $[a_{ij}]$ where $a_{ij} = \cos\left(\frac{(i+j)2\pi}{n}\right)$. Find determinant A.

Sol. $$\Delta = \begin{vmatrix} \cos\frac{4\pi}{n} & \cos\frac{6\pi}{n} & \ldots\ldots & \cos\frac{(n+1)2\pi}{n} \\ \cos\frac{6\pi}{n} & \cos\frac{8\pi}{n} & \ldots\ldots & \cos\frac{(n+2)2\pi}{n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \cos\frac{(n+1)2\pi}{n} & \cos\frac{(n+2)2\pi}{n} & \ldots\ldots & \cos\frac{(n+n)2\pi}{n} \end{vmatrix} = \begin{vmatrix} \sum_{j=1}^{n}\cos(j+1)\frac{2\pi}{n} & \cos\frac{6\pi}{n} & \ldots\ldots & \cos\frac{(n+1)2\pi}{n} \\ \sum_{j=1}^{n}\cos(j+2)\frac{2\pi}{n} & \cos\frac{8\pi}{n} & \ldots\ldots & \cos\frac{(n+2)2\pi}{n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \sum_{j=1}^{n}\cos(j+n)\frac{2\pi}{n} & \cos\frac{(n+2)2\pi}{n} & \ldots\ldots & \cos\frac{(n+n)2\pi}{n} \end{vmatrix}$$

(Applying $C_1 \to C_1 + C_2 + \ldots + C_n$) Now, $\sum_{j=1}^{n}\cos(j+1)\frac{2\pi}{n} = \sum_{j=1}^{n}\cos(j+2)\frac{2\pi}{n} = \ldots = \sum_{j=1}^{n}\cos(j+n)\frac{2\pi}{n}$

$= 1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos 2(n-1)\frac{\pi}{n} = 0$ $\Rightarrow$ value of determinant is zero.

H. MULTIPLICATION OF TWO DETERMINANTS

(i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$

Similarly two determinants of order three are multiplied.

(ii) If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ then , $D^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ where A_i , B_i , C_i are cofactors

Proof : Consider $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix}$

Note : $a_1A_2 + b_1B_2 + c_1C_2 = 0$ etc.

therefore , $D \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^3 \Rightarrow \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^2$ **or** $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ CA_3 & B_3 & C_3 \end{vmatrix} = D^2$

Ex.22 Prove that $\Delta \equiv \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

Sol. Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & (a-b)(c-d) & (a^2-b^2)(c^2-d^2) \\ 1 & (a-c)(b-d) & (a^2-c^2)(b^2-d^2) \end{vmatrix} = \begin{vmatrix} (a-b)(c-d) & (a-b)(a+b)(c-d)(c+d) \\ (a-c)(b-d) & (a-c)(a+c)(b-d)(b+d) \end{vmatrix}$$

$= (a-b)(c-d)(a-c)(b-d)\begin{vmatrix} 1 & (a+b)(c+d) \\ 1 & (a+c)(b+d) \end{vmatrix}$

$= (a-b)(c-d)(a-c)(b-d)[(a+c)(b+d) - (a+b)(c+d)]$

$= (a-b)(c-d)(a-c)(b-d)(ab+cd-ac-bd) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

Alternatively : Let $\left.\begin{matrix} bc+ad = x \\ ca+bd = y \\ ab+cd = z \end{matrix}\right]$ and using $c_3 \to c_3 + 2\,abcd\,.\,c_3$

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$$

Ex.23 Show that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-y^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$ (where $r^2 = x^2 + y^2 + z^2$ & $u^2 = xy + yz + zx$)

Sol. Consider the determinant , $\Delta = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$ We see that the L.H.S. determinant has its constituents which are the co-factor of Δ . Hence L.H.S. determinant

$$= \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xy+yz+zx \\ xy+yz+zx & y^2+z^2+x^2 & yz+zx+xy \\ zx+xy+yz & yz+xz+xy & z^2+x^2+y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$$

Ex.24 Without expanding, as for as possible, prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

Sol. Let $D = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$ for x = y, D = 0 (since C_1 and C_2 are identical)

Hence (x – y) is a factor of D (y – z) and (z – x) are factors of D. But D is a homogeneous expression of the 4th degree is x, y, z.
$\therefore$ There must be one more factor of the 1st degree in x, y, z say k (x + y + z) where k is a constant.
Let D = k (x – y) (y – z) (z – x) (x + y + z),Putting x = 0, y = 1, z = 2

then $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 8 \end{vmatrix} = k(0-1)(1-2)(2-0)(0+1+2)$

$\Rightarrow$ L(8 – 2) = k(–1) (–1) (2) (3) $\quad\therefore$ k = 1 $\quad\therefore$ D = (x – y) (y – z) (z – x) (x + y + z)

Ex.25 Prove that $\begin{vmatrix} x_1 & y_1 & 1 \\ ax_1+bx_2+cx_3 & ay_1+by_2+cy_3 & a+b+c \\ -ax_1+bx_2+cx_3 & -ay_1+by_2+cy_3 & -a+b+c \end{vmatrix} = 0.$

Sol. Given that $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ ax_1+bx_2+cx_3 & ay_1+by_2+cy_3 & a+b+c \\ -ax_1+bx_2+cx_3 & -ay_1+by_2+cy_3 & -a+b+c \end{vmatrix} = 0$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ -a & b & c \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \times 0 = 0.$$

Ex.26 Express $\begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$ as product of two determinants.

Sol. The given determinant is = $\begin{vmatrix} 1+2ax+a^2x^2 & 1+2ay+a^2y^2 & 1+2az+a^2z^2 \\ 1+2bx+b^2x^2 & 1+2by+b^2y^2 & 1+2bz+b^2z^2 \\ 1+2cx+c^2x^2 & 1+2cy+c^2y^2 & 1+2cz+c^2z^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \times \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$, with the help of row–by–row multiplication rule.

Ex.27 Let D = $\begin{vmatrix} 2a_1b_1 & a_1b_2+a_2b_1 & a_1b_3+a_3b_1 \\ a_1b_2+a_2b_1 & 2a_2b_2 & a_2b_3+a_3b_2 \\ a_1b_3+a_3b_1 & a_3b_2+a_2b_3 & 2a_3b_3 \end{vmatrix}$. Express the determinant D as a product of two determinants.

Hence or otherwise show that D = 0.

Sol. We have D = $\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix}$, as can be seen by applying row–by–row multiplication rule.

Hence D = 0.

Ex.28 If $f(x, y) = x^2 + y^2 - 2xy$, $(x, y \in R)$ and $A = \begin{bmatrix} f(x_1,y_1) & f(x_1,y_2) & f(x_1,y_3) \\ f(x_2,y_1) & f(x_2,y_2) & f(x_2,y_3) \\ f(x_3,y_1) & f(x_3,y_2) & f(x_3,y_3) \end{bmatrix}$ such that tr(A) = 0, then prove that det (A) $\geq$ 0.

Sol. tr (A) = 0 $\Rightarrow (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 0 \Rightarrow x_1 = y_1, x_2 = y_2$ and $x_3 = y_3$

Now det (A) = $\Delta = \begin{vmatrix} x_1^2+y_1^2-2x_1y_1 & x_1^2+y_2^2-2x_1y_2 & x_1^2+y_3^2-2x_1y_3 \\ x_2^2+y_1^2-2x_2y_1 & x_2^2+y_2^2-2x_2y_2 & x_2^2+y_3^2-2x_2y_3 \\ x_3^2+y_1^2-2x_3y_1 & x_3^2+y_2^2-2x_3y_2 & x_3^2+y_3^2-2x_3y_3 \end{vmatrix}$

or $\Delta = \begin{vmatrix} x_1^2 & -2x_1 & 1 \\ x_2^2 & -2x_2 & 1 \\ x_3^2 & -2x_3 & 1 \end{vmatrix} \begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix} = 2 \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} \begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$

$= 2(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)(y_1 - y_2)(y_2 - y_3)(y_3 - y_1) = 2(x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2 \geq 0$

Hence det (A) $\geq$ 0

I. SYSTEM OF LINEAR EQUATIONS

System Of Linear Equation (In Two Variables) :

(i) Consistent Equations : Definite & unique solution . [intersecting lines]
(ii) Inconsistent Equation : No solution . [Parallel line]
(iii) Dependent equation : Infinite solutions . [Identical lines]

Let $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ then

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent & $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are dependent

Cramer's Rule :[Simultaneous Equations Involving Three Unknowns]

Let $a_1x + b_1y + c_1z = d_1$ ..... (I) ; $a_2x + b_2y + c_2z = d_2$ ..... (II) ; $a_3x + b_3y + c_3z = d_3$ ..... (III)

Then , $x = \frac{D_1}{D}, Y = \frac{D_2}{D}, Z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note :

(a) If $D \neq 0$ and atleast one of $D_1 , D_2 , D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solutions.

(d) If $D = 0$ but atleast one of D_1 , D_2 , D_3 is not zero then the equations are inconsistent and have no solution.

(e) If x , y , z are not all zero , the condition for $a_1x + b_1y + c_1z = 0$; $a_2x + b_2y + c_2z = 0$ & $a_3x + b_3y + c_3z = 0$ to be consistent in x , y , z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then the given equations are said to have **Trivial Solution.**

Solving System of Linear Equations Using Matrices :

Consider the system
$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$...$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_n.$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ ..... \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ ..... \\ ..... \\ b_n \end{bmatrix}$

Then the above system can be expressed in the matrix form as AX = B.
The system is said to be consistent if it has atleast one solution.

(i) System of Linear Equations And Matrix Inverse :

If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix. If A is non–singular, solution is given by $X = A^{-1}B$.

If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinitely many solutions.

If A is singular and (adj A) B $\neq$ 0, then the system has no solution (we say it is inconsistent).

(ii) Homogeneous System and Matrix Inverse :

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is AX = O. ($\because$ in this case $b_1 = b_2 = \ldots\ldots b_n = 0$), where A is a square matrix.

If A is non–singular, the system has only the trivial solution (zero solution) X = 0

If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non–trivial solutions.

(iii) Elementary Row Transformation of Matrix :

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(c) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

Remark : Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write A **~** B.

(iv) Echelon Form of A Matrix : A matrix is said to be in Echelon form if it satisfies the following

(a) The first non-zero element in each row is 1 & all the other elements in the corresponding column (i.e. the column where 1 appears) are zeroes.

(b) The number of zeros before the first non zero element in any non zero row is less than the number of such zeroes in succeeding non zero rows.

(v) System of Linear Equations : Let the system be AX = B where A is an m × n matrix, X is the n–column vector & B is the m–column vector. Let [AB] denote the **augmented matrix** (i.e. matrix obtained by accepting elements of B as n + 1[th] column & first n columns are that of A).

Ex.29 Solve the equations $\lambda x + 2y - 2z - 1 = 0$,

$4x + 2\lambda y - z - 2 = 0$,

$6x + 6y + \lambda z - 3 = 0$, considering specially the case when $\lambda = 2$.

Sol. The matrix form of the given system is $\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .....(i)

The given system of equations will have a unique solution if and only if the coefficient matrix is non–singular, i.e., iff $\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0$ i.e., iff $\lambda^3 + 11\lambda - 30 \neq 0$ i.e., iff $(\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 0$.

Now the only real root of the equation $(\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 2 = 0$ is $\lambda = 2$

Therefore if $\lambda \neq 2$, the given system of equations will have a unique solution given by

$$\frac{x}{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 2\lambda & -1 \\ 3 & 6 & \lambda \end{vmatrix}} = \frac{y}{\begin{vmatrix} \lambda & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & \lambda \end{vmatrix}} = \frac{z}{\begin{vmatrix} \lambda & 2 & 1 \\ 4 & 2\lambda & 2 \\ 6 & 6 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}$$

In case $\lambda = 2$, the equation (i) becomes $\begin{bmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Performing $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get $\begin{bmatrix} 2 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The above system of equations is equivalent to $8z = 0$, $3z = 0$, $2x + 2y - 2z = 1$.

$\therefore$ $x = \frac{1}{2} - c$, $y = c$, $z = 0$ constitute the general solution of the given system of equations in case $\lambda = 2$.

Ex.30 Solve
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 10 \end{aligned}$$

Sol.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 10 \end{aligned} \xrightarrow[-7E1+E3]{04E1+E2} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ -3x_2 - 6x_3 &= -9 \\ -6x_2 - 12x_3 &= -18 \end{aligned}$$

$$\xrightarrow{-2E_2+E3} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ -3x_2 - 6x_3 &= -9 \\ 0 &= 0 \end{aligned} \xrightarrow{-\frac{1}{3}E_2} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 - 2x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

Now we have only two equations in three unknowns. In the second equation, we can let $x_3 = k$, where k is any complex number. Then $x_2 = 3 - 2k$. Substituting $s_3 = k$ and $x_2 = 3 - 2k$ into the first equation, we have $x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(3 - 2k) - 3(k) = -2 + k$

Thus the general solution is $(-2 + k, 3 - 2k, k)$ or
$$\begin{aligned} x_1 &= -2 + k \\ x_2 &= 3 - 2k \\ x_3 &= k \end{aligned}$$

And we see that the system has an infinite number of solutions. Specific solutions can be generated by choosing specific values for k.

Ex.31 Number of triplets of a, b & c for which the system of equations $ax - by = 2a - b$ and $(c + 1)x + cy = 10 - a + 3b$ has infinitely many solutions and $x = 1$, $y = 3$ is one of the solutions is

Sol. put $x = 1$ & $y = 3$ in 1st equation $\Rightarrow a = -2b$ & from 2nd equation

$c = \frac{9 + 5b}{4}$; Now use $\frac{a}{c+1} = -\frac{b}{c} = \frac{2a - b}{10 - a + 3b}$; from first two $b = 0$ or $c = 1$;

if $b = 0 \Rightarrow a = 0$ & $c = 9/4$; if $c = 1$; $b = -1$; $a = 2$

Ex.32 Solve $\begin{array}{r} x_1 + 2x_2 + 3x_3 = 4 \\ 4x_1 + 5x_2 + 6x_3 = 7 \\ 7x_1 + 8x_2 + 9x_3 = 12 \end{array}$

Sol. $\begin{array}{r} x_1 + 2x_2 + 3x_3 = 4 \\ 4x_1 + 5x_2 + 6x_3 = 7 \\ 7x_1 + 8x_2 + 9x_3 = 12 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 12 \end{array}\right) \xrightarrow[-7E2+E3]{-4E1+E2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -12 & -16 \end{array}\right)$

$$\xrightarrow{-2E2+E3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 2 \end{array}\right) \quad \begin{array}{r} x_1 + 2x_2 + 3x_2 = \;\; 4 \\ 0x_1 - 3x_2 - 6x_3 = -9 \\ 0x_1 + 0x_2 + 0x_3 = \;\; 2 \end{array}$$

The last equation, 0 = 2, can never hold regardless of the values assigned to x_1, x_2 and x_3. Because the last (equivalent) system has no solution, the original system of equations has no solution.

Ex.33 Solve $\begin{array}{r} x_2 - x_3 = -9 \\ 2x_1 - x_2 + 4x_3 = 29 \\ x_1 + x_2 - 3x_3 = -20 \end{array}$ by reducing the augmented matrix of the system to reduced row echelon form.

Sol. $\left(\begin{array}{ccc|c} 0 & 1 & -1 & -9 \\ 2 & -1 & 4 & 29 \\ 1 & 1 & -3 & -20 \end{array}\right) \xrightarrow{R1 \leftrightarrow R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 2 & -1 & 4 & 29 \\ 0 & 1 & -1 & -9 \end{array}\right) \xrightarrow{-2R1+R2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & -3 & 10 & 69 \\ 0 & 1 & -1 & -9 \end{array}\right) \xrightarrow{-\frac{1}{3}R2}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 1 & -1 & -9 \end{array}\right) \xrightarrow{-1R2+R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 0 & \frac{7}{3} & 14 \end{array}\right) \xrightarrow{\frac{3}{7}R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 0 & 1 & 6 \end{array}\right) \xrightarrow[\substack{3R3+R1 \\ -1R2+R1}]{\frac{10}{3}R3+R2}$$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 6 \end{array}\right)$. It is easy to see that $x_1 = 1$, $x_2 = -3$, $x_3 = 6$. The process of solving a system by reducing the augmented matrix to reduced row echelon form is called Gauss–Jordan elimination.

Ex.34 Determine conditions on a, b and c so that $\begin{array}{r} x_1 + 2x_2 + 3x_3 = a \\ 4x_1 + 5x_2 + 6x_3 = b \\ 7x_1 + 8x_2 + 9x_3 = c \end{array}$

will have no solutions or have an infinite number of solution.

Sol. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -3 & -6 & b-4a \\ 0 & 0 & 0 & c-2b+a \end{array}\right)$. If c – 2b + a ≠ 0, then no solution exists. If c – 2b + a = 0, we have two equations in three unknowns and we can set x_3 arbitrarily and then solve for x_1 and x_2.

Ex.35 Solve
$$\begin{aligned} x_1+2x_2+3x_3&=0\\ 4x_1+5x_2+6x_3&=0\\ 7x_1+8x_2+9x_3&=0\\ 10x_1+11x_2+12x_3&=0 \end{aligned}$$

Sol. Using Gaussian elimination with the augmented matrix.

$$\left(\begin{array}{ccc|c} 1&2&3&0\\4&5&6&0\\7&8&9&0\\10&11&12&0\end{array}\right) \xrightarrow[-10E1+E4]{\substack{-4E1+E2\\-7E1+E3}} \left(\begin{array}{ccc|c} 1&2&3&0\\0&-3&-6&0\\0&-6&-12&0\\0&-9&-18&0\end{array}\right) \xrightarrow{-\frac{1}{3}E2} \left(\begin{array}{ccc|c} 1&2&3&0\\0&1&2&0\\0&-6&-12&0\\0&-9&-18&0\end{array}\right) \xrightarrow[9E2+E2]{6E2+E3} \left(\begin{array}{ccc|c} 1&2&3&0\\0&1&2&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$

Therefore $\begin{aligned} x_1+2x_2+3x_3&=0\\ x_2+2x_3&=0\end{aligned}$ and setting $x_3 = k$ gives $\begin{aligned} x_2&=-2k\\ x_1&=-2x_2-3x_3=4k-3k=k\end{aligned}$

So we have $\begin{aligned} x_1&=k\\ x_2&=-2k\\ x_3&=k\end{aligned}$

J. INVERSE OF A MATRIX

(i) Singular & Non Singular Matrix : A square matrix A is said to be singular or non–singular according as |A| is zero or non–zero respectively.

Ex.36 Show that every skew-symmetric matrix of odd order is singular.

Sol. Since $|A| = |A'| = (-1)^n |A| \Rightarrow |A| (1 - (-1)^n) = 0$.
As n is odd $\Rightarrow |A| = 0$. Hence A is singular.

(ii) Cofactor Matrix & Adjoint Matrix : Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.
i.e. If $A = [a_{ij}]_n$ then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of $a_{ij} \;\forall$ i & j.
Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji} \;\forall$ i & j.

(iii) Properties of Cofactor A and adj A :

(a) A . adj A = $|A|\, I_n$ = (adj A) A where $A = [a_{ij}]_n$.
(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A. In particular, for 3 × 3 matrix, $|\text{adj } A| = |A|^2$
(c) If A is a symmetric matrix, then adj A are also symmetric matrices.
(d) If A is singular, then adj A is also singular.

(iv) Inverse of A Matrix (Reciprocal Matrix) : Let A be a non–singular matrix. Then the matrix $\frac{1}{|A|}$ adj A is the multiplicative inverse of A(we call it inverse of A) and is denoted by A^{-1}.

When have A (adj A) = $|A|\, I_n$ = (adj A) A

$$\Rightarrow A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right)A, \text{ for A is non–singular } \Rightarrow A^{-1} = \frac{1}{|A|}\text{ adj } A.$$

Remarks :

1. The necessary and sufficient condition for existence of inverse of A is that A is non–singular.
2. A^{-1} is always non–singular.
3. If A = diag $(a_{11}, a_{22}, \ldots, a_{nn})$ where $a_{ij} \neq 0 \ \forall$ i, then A^{-1} = diag $(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$.
4. $(A^{-1})' = (A')^{-1}$ for any non–singular matrix A. Also adj (A') = (adj A)$'$.
5. $(A^{-1})^{-1} = A$ if A is non–singular.
6. Let k be non–zero scalar & A be a non–singular matrix. Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.
7. $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$
8. Let A be a non–singular matrix. Then AB = AC $\Rightarrow$ B = C & BA = CA $\Rightarrow$ B = C.
9. A is non–singular and symmetric $\Rightarrow A^{-1}$ is symmetric.
10. In general AB = **0** does not imply A = **0** or B = **0**. But if **A** is non–singular and **AB = 0,** then **B = 0**. Similarly B is non–singular and **AB = 0** $\Rightarrow$ **A = 0**. Therefore, **AB = 0** $\Rightarrow$ either both are singular or one of them is **0**.

Characteristic Polynomial & Characteristic Equation : Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called as characteristic equation A.

Remark : Every square matrix A satisfies its characteristic equation (Cayley – Hamilton Theorem).
i.e. $a_0 x^n + a, n^{n-1} + \ldots\ldots + a_{n-1} x + a_n = 0$ is the characteristic equation of A, then
$a_0 A^n + a_1 A^{n-1} + \ldots\ldots + a_{n-1} A + a_n I = 0$

Ex.37 Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$.

Sol. We have $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{vmatrix}$. The cofactors of the elements of the first row of | A | are

$\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix}$ i.e., are 15, 0, – 10 respectively.

The cofactors of the elements of the second row of |A| are $-\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

i.e. are 6, –3, 0 respectively.

The cofactors of the elements of the third row of |A| are $\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}$

i.e., are – 15, 0, 5 respectively.

Therefore the adj. A = the transpose of the matrix B where $B = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$.

$\therefore$ adj $A = \begin{bmatrix} 15 & 0 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

Ex.38 If A and B are square matrices of the same order, then adj (AB) = adj B. adj A.

Sol. We have AB adj (AB) = |AB| I_n = (adj AB) AB. ...(1)

Also AB (adj B. adj A) = A(B adj B) adj A

= A |B| I_n adj A = |B| (A adj A) = | B | | A | I_n = | BA | I_n = | AB | I_n. ...(2)

Similarly, we have (adj B adj A) AB = adj B [(adj A [(adj A) A] B

= adj B. |A| I_n B = |A|. (adj B) B = |A|. |B| I_n = | AB| I_n. ...(3)

From (1), (2) and (3), the required result follows, provided AB is non–singular.

Note : The result adj (AB) = adj B adj A holds goods even if A or B is singular. However the proof given above is applicable only if A and B are non–singular.

Ex.39 If A be an n-square matrix and B be its adjoint, then show that Det (AB + KI_n) = [Det (A) + K]^n, where K is a scalar quantity.

Sol. We have, A B = A (adj A) = Det (A). I_n $\Rightarrow$ A B + K I_n = Det (A) I_n + K I_n

$\Rightarrow$ Det (A B + K I_n) = Det (Det (A) I_n + K I_n) = (Det (A) + K)^n ($\because$ Det (α I_n) = α^n)

$\Rightarrow$ Det (A B + K I_n) = [Det (A) + K]^n.

Ex.40 If (ℓ_r, m_r, n_r), r = 1, 2, 3 be the direction cosines of three mutually perpendicular lines referred to an orthogonal Cartesian co–ordinate system, then prove that $\begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}$ is an orthogonal matrix.

Sol. Let A = $\begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}$. Then A′ = $\begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$. We have AA′ = $\begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}\begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$.

$$= \begin{bmatrix} \ell_1^2 + m_1^2 + n_1^2 & \ell_1\ell_2 + m_1m_2 + n_1n_2 & \ell_1\ell_3 + m_1m_2 + n_1n_3 \\ \ell_2\ell_1 + m_2m_1 + n_2n_1 & \ell_2^2 + m_2^2 + n_2^2 & \ell_2\ell_3 + m_2m_3 + n_2n_3 \\ \ell_3\ell_1 + m_3m_1 + n_3n_1 & \ell_3\ell_2 + m_3m_2 + n_3n_2 & \ell_3^2 + m_3^2 + n_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

$\begin{bmatrix} \therefore & \ell_1^2 + m_1^2 + n_1^2 = 1 \text{ etc.} \\ \text{and} & \ell_1\ell_2 + m_1m_2 + n_2n_3 = 0 \text{ etc.} \end{bmatrix}$ Hence the matrix A is orthogonal.

Ex.41 Obtain the characteristic equation of the matrix A = $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and verify that it is satisfied y A and hence find its inverse.

Sol. We have |A – λI| = $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix}$

= (1 – λ) (2 – λ) (3 – λ) + 2[0 – 2(2 – λ)] = (2 – λ) [(1 – λ) (3 – λ) – 4]

= (2 – λ) [λ^2 – 4λ – 1] = –(λ^3 – 6λ^2 + 7λ + 2).

$\therefore$ the characteristic equation of A is $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$ ......(i)

By the Cayley–Hamilton theorem $A^3 - 6A^2 + 7A + 2I = O$. .....(ii)

Verification of (ii). We have $A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$.

Also $A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$.

Now $A^2 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$.

Hence Cayley–Hamilton theorem is verified. Now we shall compute A^{-1}.
Multiplying (ii) by A^{-1}, we get $A^2 - 6A + 7I + 2A^{-1} = \mathbf{O}$.

$$\therefore \quad A^{-1} = -\frac{1}{2}(A^2 - 6A + 7I) = -\frac{1}{2}\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 3 & 13 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}.$$

Ex.42 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol. We have $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}$, applying $C_3 \to C_3 \to 2C_2 = -1\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}$, expanding the determinant along the first row $= -2$. Since $|A| \neq 0$, therefore A^{-1} exists.

Now the cofactors of the elements of the first row of $|A|$ are $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

i.e., are $-1, 8, -5$ respectively.

The cofactors of the elements of the second row of $|A|$ are $-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$

i.e. are $1, -6, 3$ respectively.

The cofactors of the elements of the third row of $|A|$ are $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

i.e. are $-1, 2, -1$ respectively.

Therefore the Adj. A = the transpose of the matrix B where $B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$ $\therefore$ Adj. $A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ 5 & 3 & -1 \end{bmatrix}$.

Now $A^{-1} = \frac{1}{|A|}$ Adj. A and here $|A| = -2$. $\quad \therefore \; A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

Ex.43 If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric.

Sol. Since A is symmetric, $A' = A \quad \Rightarrow \quad A'A^{-1} = AA^{-1} = I$

$\Rightarrow \quad (A^{-1})A'A^{-1} = (A^{-1})'I = (A^{-1})'I' \quad \Rightarrow \quad (AA^{-1})'A^{-1} = (A^{-1}I)' = (A^{-1})'$

$\Rightarrow \quad I'A^{-1} = (A^{-1})' \Rightarrow A^{-1} = (A^{-1})'$. Hence A^{-1} is also symmetric.